

WELL-POISED GENERATION OF APÉRY-LIKE RECURSIONS

WADIM ZUDILIN[‡] (Moscow)

E-print math.NT/0307058

May/November 2003

ABSTRACT. The idea to use classical hypergeometric series and, in particular, well-poised hypergeometric series in diophantine problems of the values of the polylogarithms has led to several novelties in number theory and neighbouring areas of mathematics. Here we present a systematic approach to derive second-order polynomial recursions for approximations to some values of the Lerch zeta function, depending on the fixed (but not necessarily real) parameter α satisfying the condition $\operatorname{Re}(\alpha) < 1$. Substituting $\alpha = 0$ into the resulting recurrence equations produces the famous recursions for rational approximations to $\zeta(2)$, $\zeta(3)$ due to Apéry, as well as the known recursion for rational approximations to $\zeta(4)$. Multiple integral representations for solutions of the constructed recurrences are also given.

The idea to use classical hypergeometric series [9, 11] and, in particular, well-poised hypergeometric series [13] in diophantine problems of the values of the polylogarithms has led to several novelties in number theory and neighbouring fields of mathematics. Here we present a systematic approach to derive second-order polynomial recursions for approximations to the numbers

$$Z_2^-(\alpha) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{(\nu-\alpha)^2}, \quad Z_3(\alpha) = \sum_{\nu=1}^{\infty} \frac{1}{(\nu-\alpha)^3}, \quad Z_4(\alpha) = \sum_{\nu=1}^{\infty} \frac{1}{(\nu-\alpha)^4},$$

where the fixed (but not necessarily real) parameter α satisfies the condition $\operatorname{Re}(\alpha) < 1$. Substituting $\alpha = 0$ into the resulting recurrence equations produces the famous recursions for rational approximations to $\zeta(2) = 2Z_2^-(0)$, $\zeta(3) = Z_3(0)$ due to Apéry [1], as well as the recursion for rational approximations to $\zeta(4) = Z_4(0)$ known as the Cohen–Rhin–Sorokin–Zudilin recursion ([8, 15, 17]), which is proper from both the historical and the alphabetic point of view.

To make clear to the reader, what do we mean by a *recursion for approximations to a number* $z \in \mathbb{C}$, we introduce a formal definition. The requirement to such the recursion is to have two linearly independent solutions $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ (uniquely determined by the recursion itself and initial conditions) such that $v_n/u_n \rightarrow z$ as

Key words and phrases. Hypergeometric series, polynomial recursion, Apéry's approximations, zeta value, multiple integral.

[‡]The work is supported by an Alexander von Humboldt research fellowship.

$n \rightarrow \infty$. In the case $z \in \mathbb{R}$ (e.g., corresponding to $\alpha \in \mathbb{Q}$ in the above definitions), we usually restrict ourselves to the sequences $\{u_n\}_{n=0}^\infty, \{v_n\}_{n=0}^\infty$ consisting of rational numbers only and may interpret them as denominators and numerators, respectively, of rational convergents to z .

We apologise in advance for facing the reader with, sometimes, cumbersome formulae. Although the ideas of the well-poised hypergeometric construction of linear forms $r_n = u_n z - v_n$, $n = 0, 1, 2, \dots$, are simple, the appearance of lengthy formulae is unavoidable in this type of analysis. Our main results are the recursions (4), (8), (10), (12), (13), and the integral representations (6), (9), (11) (generalizing those of [3]) for the linear forms $\{r_n\}_{n=0}^\infty$.

1. $Z_2^-(\alpha)$. Take the rational function

$$\begin{aligned} R_n(t) &= n! \cdot (2t - 2\alpha + n) \frac{(t-1) \cdots (t-n) \cdot (t-2\alpha+n+1) \cdots (t-2\alpha+2n)}{((t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n))^3} \\ &= 2n! \cdot \frac{\Gamma(t-\alpha+n/2+1)}{\Gamma(t-\alpha+n/2)} \cdot \frac{\Gamma(t)}{\Gamma(t-n)} \\ &\quad \times \left(\frac{\Gamma(t-\alpha)}{\Gamma(t-\alpha+n+1)} \right)^3 \cdot \frac{\Gamma(t-2\alpha+2n+1)}{\Gamma(t-2\alpha+n+1)} \end{aligned}$$

satisfying the property

$$t \mapsto -(t-2\alpha+n) : \quad R_n(t) \mapsto (-1)^n R_n(t). \quad (1)$$

Decompose the function $R_n(t)$ as the sum of partial fractions,

$$R_n(t) = \sum_{j=0}^2 \sum_{k=0}^n \frac{A_{jk}(n)}{(t-\alpha+k)^{3-j}},$$

and use the last representation as in [19], proof of Lemma 1, to sum the quantity

$$\begin{aligned} r_n &= \sum_{t=1}^{\infty} (-1)^{t-1} R_n(t) = \sum_{t=n+1}^{\infty} (-1)^{t-1} R_n(t) \\ &= u_{0n} \sum_{\nu=1}^{\infty} \frac{(-1)^{n-1}}{(\nu-\alpha)^3} + u_{1n} \sum_{\nu=1}^{\infty} \frac{(-1)^{n-1}}{(\nu-\alpha)^2} + u_{2n} \sum_{\nu=1}^{\infty} \frac{(-1)^{n-1}}{\nu-\alpha} - v_n, \end{aligned} \quad (2)$$

where

$$u_{jn} = \sum_{k=0}^n (-1)^k A_{jk}(n), \quad j = 0, 1, 2, \quad v_n = \sum_{j=0}^2 \sum_{k=0}^n (-1)^n A_{jk}(n) \sum_{\nu=1}^k \frac{(-1)^{\nu-1}}{(\nu-\alpha)^{3-j}}.$$

The property (1) yields $u_{0n} = u_{2n} = 0$ for all $n = 0, 1, 2, \dots$, hence setting $u_n = u_{1n}/2$ we obtain the linear forms

$$r_n = u_n \cdot 2Z_2^-(\alpha) - v_n \in \mathbb{Q}Z_2^-(\alpha) + \mathbb{Q}, \quad n = 0, 1, 2, \dots, \quad (3)$$

with effectively determined coefficients u_n and v_n . Applying Zeilberger's creative telescoping [12] in the manner of [19], Section 2 (namely, computing the certificate and the corresponding difference annihilating operator for the function $(-1)^t R_n(t)$, which is rational with respect to either t or n), we arrive at the following recursion satisfied by both the linear forms (3) and their coefficients:

$$\begin{aligned}
& (n+1)^2(n+1-\alpha)^2(5n^2-4\alpha n+\alpha^2)u_{n+1} \\
& - (55n^6-11(14\alpha-15)n^5+(179\alpha^2-385\alpha+180)n^4 \\
& - (116\alpha^3-358\alpha^2+332\alpha-85)n^3+(45\alpha^4-174\alpha^3+232\alpha^2-113\alpha+15)n^2 \\
& - \alpha(\alpha-1)(10\alpha^3-35\alpha^2+41\alpha-12)n+\alpha^2(\alpha-1)^2(\alpha^2-3\alpha+3))u_n \\
& - n^2(n-\alpha)^2(5(n+1)^2-4\alpha(n+1)+\alpha^2)u_{n-1} = 0, \\
& u_0 = 1, \quad u_1 = \alpha^2 - 3\alpha + 3, \\
& v_0 = 0, \quad v_1 = \frac{\alpha^2 - 4\alpha + 5}{(\alpha - 1)^2}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{u_n} = 2Z_2^-(\alpha).
\end{aligned} \tag{4}$$

(We justify the latter limit relation by notifying the following consequence of the forthcoming formula (6): $r_n \rightarrow 0$ as $n \rightarrow \infty$.)

On the other hand, the series (2) may be easily identified with the very-well-poised hypergeometric ${}_6F_5(-1)$ -series

$$\begin{aligned}
r_n &= (-1)^n n! \sum_{\nu=0}^{\infty} (3n+2-2\alpha+2\nu) \\
& \times \frac{\Gamma(3n+2-2\alpha+\nu) \Gamma(n+1-\alpha+\nu)^3 \Gamma(n+1+\nu)}{\Gamma(1+\nu) \Gamma(\nu+n+1-\alpha)^3 \Gamma(2n+2-2\alpha+\nu)} (-1)^\nu,
\end{aligned} \tag{5}$$

which admits the double integral representation

$$r_n = u_n \cdot 2Z_2^-(\alpha) - v_n = (-1)^n \iint_{[0,1]^2} \frac{x^{n-\alpha}(1-x)^n y^{n-\alpha}(1-y)^n}{(1-x(1-y))^{n+1}} dx dy \tag{6}$$

(see [17], Theorem 5). Whipple's transformation ([2], Section 4.4, formula (2)) gives one a more direct way to deduce the integral (6): first convert the series (5) into the hypergeometric ${}_3F_2(1)$ -series

$$r_n = \Gamma(n+1-\alpha) \sum_{\nu=0}^{\infty} \frac{\Gamma(n+1+\nu)^2 \Gamma(n+1-\alpha+\nu)}{\Gamma(1+\nu) \Gamma(2n+2-\alpha+\nu)^2}$$

and secondly use the Euler-type integral formula for the latter series.

2. $Z_3(\alpha)$. This time, take the rational function

$$R_n(t) = n!^2 \cdot (2t-2\alpha+n) \frac{(t-1) \cdots (t-n) \cdot (t-2\alpha+n+1) \cdots (t-2\alpha+2n)}{((t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n))^4}$$

satisfying the property

$$t \mapsto -(t - 2\alpha + n) : \quad R_n(t) \mapsto -R_n(t). \quad (7)$$

After partial-fraction decomposition we arrive at the quantity

$$r_n = \frac{1}{2} \sum_{t=1}^{\infty} R_n(t) = u_n Z_3(\alpha) - v_n \in \mathbb{Q} Z_3(\alpha) + \mathbb{Q}, \quad n = 0, 1, 2, \dots,$$

with effectively computable coefficients u_n and v_n . Zeilberger's creative telescoping produces the recursion

$$\begin{aligned} & (n+1)^3(n+1-\alpha)^3(2n-\alpha)(3n^2-3\alpha n+\alpha^2)u_{n+1} \\ & - (2n+1-\alpha)(102n^8-408(\alpha-1)n^7+2(359\alpha^2-714\alpha+321)n^6 \\ & - 6(\alpha-1)(121\alpha^2-238\alpha+83)n^5+3(152\alpha^4-605\alpha^3+811\alpha^2-415\alpha+64)n^4 \\ & - 2(\alpha-1)(89\alpha^4-367\alpha^3+461\alpha^2-177\alpha+15)n^3 \\ & + \alpha(40\alpha^5-267\alpha^4+634\alpha^3-669\alpha^2+304\alpha-45)n^2 \\ & - \alpha^2(\alpha-1)(2\alpha-1)(2\alpha^3-17\alpha^2+37\alpha-25)n - \alpha^3(\alpha-1)^2(2\alpha^2-6\alpha+5))u_n \\ & + n^3(n-\alpha)^3(2(n+1)-\alpha)(3(n+1)^2-3\alpha(n+1)+\alpha^2)u_{n-1} = 0, \quad (8) \\ & u_0 = 1, \quad u_1 = 2\alpha^2 - 6\alpha + 5, \\ & v_0 = 0, \quad v_1 = \frac{(\alpha^2 - 3\alpha + 3)(\alpha - 2)}{(\alpha - 1)^3}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{u_n} = Z_3(\alpha), \end{aligned}$$

while writing r_n as a very-well-poised hypergeometric ${}_8F_7(1)$ -series and applying [17], Theorem 5, we obtain the triple integral

$$r_n = u_n Z_3(\alpha) - v_n = \frac{1}{2} \iiint_{[0,1]^3} \frac{x^{n-\alpha}(1-x)^n y^{n-\alpha}(1-y)^n z^{n-\alpha}(1-z)^n}{(1-x(1-y(1-z)))^{n+1}} dx dy dz. \quad (9)$$

Bailey's transformation ([2], Section 6.3, formula (2)) allows us to write the quantity r_n as the Barnes-type integral

$$\begin{aligned} r_n &= \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(n+1+s)^2 \Gamma(n+1-\alpha+s)^2 \Gamma(-s)^2}{\Gamma(2n+2-\alpha+s)^2} ds \\ &= -\frac{1}{2} \sum_{t=1}^{\infty} \frac{d}{dt} \left(\frac{(t-1)(t-2) \cdots (t-n)}{(t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n)} \right)^2, \end{aligned}$$

where the real constant c lies in the interval $\operatorname{Re}(\alpha) - 1 < c < 0$. This gives another way to deduce the recursion (8), by applying Zeilberger's creative telescoping directly to the latter summation (cf. [18], Lemmas 1–3, for the proof in the particular case $\alpha = 0$).

3. $Z_4(\alpha)$. Finally, take the rational function

$$R_n(t) = (2t - 2\alpha + n) \left(\frac{(t-1) \cdots (t-n) \cdot (t-2\alpha+n+1) \cdots (t-2\alpha+2n)}{((t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n))^2} \right)^2$$

satisfying the property (7) and consider the quantity

$$r_n = -\frac{(-1)^n}{2} \sum_{t=1}^{\infty} \frac{dR_n(t)}{dt} = u_n \cdot 6Z_4(\alpha) - v_n \in \mathbb{Q}Z_4(\alpha) + \mathbb{Q}, \quad n = 0, 1, 2, \dots$$

Then Zeilberger's creative telescoping gives the recursion

$$\begin{aligned} & (n+1)^5(n+1-\alpha)^3(n+1-2\alpha)(39n^4-65\alpha n^3+45\alpha^2n^2-15\alpha^3n+2\alpha^4)u_{n+1} \\ & - (10530n^{13}-1755(40\alpha-39)n^{12}+18(11881\alpha^2-23400\alpha+10881)n^{11} \\ & - 9(43964\alpha^3-130691\alpha^2+122499\alpha-36036)n^{10} \\ & + (497482\alpha^4-1978380\alpha^3+2795153\alpha^2-1651455\alpha+343161)n^9 \\ & - (449452\alpha^5-2238669\alpha^4+4229444\alpha^3-3756546\alpha^2+1559025\alpha-241137)n^8 \\ & + 2(149999\alpha^6-898904\alpha^5+2128142\alpha^4-2523748\alpha^3 \\ & + 1567577\alpha^2-480285\alpha+56394)n^7 \\ & - (149336\alpha^7-1049993\alpha^6+2995163\alpha^5-4449872\alpha^4 \\ & + 3679649\alpha^3-1676024\alpha^2+385125\alpha-33930)n^6 \\ & + (55088\alpha^8-448008\alpha^7+1503025\alpha^6-2693161\alpha^5+2786514\alpha^4 \\ & - 1681907\alpha^3+568968\alpha^2-96291\alpha+5967)n^5 \\ & - (14696\alpha^9-137720\alpha^8+536294\alpha^7-1132580\alpha^6+1413762\alpha^5 \\ & - 1065166\alpha^4+474344\alpha^3-116539\alpha^2+13455\alpha-468)n^4 \\ & + \alpha(\alpha-1)(2692\alpha^8-26700\alpha^7+105832\alpha^6-220076\alpha^5+260191\alpha^4 \\ & - 176174\alpha^3+65540\alpha^2-11955\alpha+780)n^3 \\ & - \alpha^2(\alpha-1)^2(304\alpha^7-3430\alpha^6+14198\alpha^5-29252\alpha^4 \\ & + 32370\alpha^3-18825\alpha^2+5265\alpha-540)n^2 \\ & + 2\alpha^3(\alpha-1)^3(8\alpha^6-128\alpha^5+581\alpha^4-1198\alpha^3+1220\alpha^2-558\alpha+90)n \\ & + 4\alpha^4(\alpha-1)^4(\alpha-2)(2\alpha-1)(\alpha^2-3\alpha+3)u_n \\ & - n^3(n-\alpha)^3(3n-2\alpha)(3n+1-2\alpha)(3n-1-2\alpha)(39n^4-13(5\alpha-12)n^3 \\ & + 3(15\alpha^2-65\alpha+78)n^2-3(5\alpha^3-30\alpha^2+65\alpha-52)n \\ & + (2\alpha^4-15\alpha^3+45\alpha^2-65\alpha+39))u_{n-1} = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} u_0 &= 1, \quad u_1 = (\alpha-1)(\alpha-2)(\alpha^2-3\alpha+3), \\ v_0 &= 0, \quad v_1 = -\frac{2\alpha^4-15\alpha^3+45\alpha^2-65\alpha+39}{(\alpha-1)^3}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{u_n} = 6Z_4(\alpha), \end{aligned}$$

and Theorem 2 in [22] yields the following 5-fold integral:

$$\begin{aligned} r_n &= u_n \cdot 6Z_4(\alpha) - v_n = \frac{(-1)^n \Gamma(3n+2-2\alpha)}{2\Gamma(n+1)\Gamma(n+1-\alpha)^2} \\ &\times \int \cdots \int_{[0,1]^5} \frac{x_1^n (1-x_1)^n \prod_{j=2}^5 x_j^{n-\alpha} (1-x_j)^n dx_1 \cdots dx_5}{(x_1(1-(1-(1-(1-x_2)x_3)x_4)x_5) + (1-x_1)x_2x_3x_4x_5)^{n+1}}. \end{aligned} \tag{11}$$

4. Other recursions. Polynomial recursions for approximations to the numbers

$$Z_1^-(\alpha) = \sum_{\nu=1}^{\infty} \frac{(-1)^{n-1}}{\nu - \alpha}, \quad Z_2(\alpha) = \sum_{\nu=1}^{\infty} \frac{1}{(\nu - \alpha)^2}$$

may be constructed by means of simpler (not well-poised) hypergeometric series. Namely, taking

$$\begin{aligned} r_n = u_n Z_1^-(\alpha) - v_n &= (-1)^n \sum_{t=1}^{\infty} (-1)^{t-1} \frac{(t-1)(t-2) \cdots (t-n)}{(t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n)} \\ &= \int_0^1 \frac{x^{n-\alpha}(1-x)^n}{(1+x)^{n+1}} dx \end{aligned}$$

we obtain the second-order recursion

$$\begin{aligned} (n+1)(n+1-\alpha)(2n-\alpha)u_{n+1} - (2n+1-\alpha)(6n^2 - 6(\alpha-1)n + \alpha(2\alpha-3))u_n \\ + n(n-\alpha)(2(n+1)-\alpha)u_{n-1} &= 0, \\ u_0 = 1, \quad u_1 = -2\alpha + 3, \quad v_0 = 0, \quad v_1 = \frac{\alpha-2}{\alpha-1}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{u_n} &= Z_1^-(\alpha), \end{aligned} \tag{12}$$

while choosing

$$\begin{aligned} r_n = u_n Z_2(\alpha) - v_n &= (-1)^n \sum_{t=1}^{\infty} \frac{n! \cdot (t-1)(t-2) \cdots (t-n)}{((t-\alpha)(t-\alpha+1) \cdots (t-\alpha+n))^2} \\ &= (-1)^n \iint_{[0,1]^2} \frac{x^{n-\alpha}(1-x)^n y^{n-\alpha}(1-y)^n}{(1-xy)^{n+1}} dx dy \end{aligned}$$

we arrive at the recursion

$$\begin{aligned} (n+1)^2(n+1-\alpha)^2(5n^2 - 6\alpha n + 2\alpha^2)u_{n+1} \\ - (55n^6 - 11(16\alpha - 15)n^5 + 2(117\alpha^2 - 220\alpha + 90)n^4 \\ - (160\alpha^3 - 468\alpha^2 + 388\alpha - 85)n^3 + (56\alpha^4 - 240\alpha^3 + 316\alpha^2 - 142\alpha + 15)n^2 \\ - 2\alpha(\alpha-1)(4\alpha^3 - 24\alpha^2 + 32\alpha - 9)n - 2\alpha^2(\alpha-1)^2(2\alpha-3))u_n \\ - n^2(n-\alpha)^2(5(n+1)^2 - 6\alpha(n+1) + 2\alpha^2)u_{n-1} &= 0, \\ u_0 = 1, \quad u_1 = -2\alpha + 3, \quad v_0 = 0, \quad v_1 = \frac{2\alpha^2 - 6\alpha + 5}{(\alpha-1)^2}, \quad \lim_{n \rightarrow \infty} \frac{v_n}{u_n} &= Z_2(\alpha). \end{aligned} \tag{13}$$

The approach presented above allows to derive a higher-order polynomial recursions for simultaneous approximations to odd and even zeta values and their α -shifts (see [21]).

5. Modular remarks. It is worth mentioning that Apéry's recursions for rational approximations to $\zeta(2)$ and $\zeta(3)$ (that correspond to the case $\alpha = 0$ in (4)

or (13) and in (8)) have a very nice modular interpretation: the generating function $U(z) = \sum_{n=0}^{\infty} u_n z^n \in \mathbb{Z}[[z]]$ becomes a modular form after substituting a suitable modular function $z = z(\tau)$ (see [4, 7]). This phenomenon happens for several other Apéry-like recursions as well (see [5, 16]); Zagier's technique in [16] allowed to construct new simple recursions for rational approximations to the numbers

$$\sum_{\nu=1}^{\infty} \frac{\left(\frac{-3}{\nu}\right)}{\nu^2} \quad \text{and} \quad \sum_{\nu=1}^{\infty} \frac{\left(\frac{-4}{\nu}\right)}{\nu^2} = \frac{1}{4} Z_2^{-}\left(\frac{1}{2}\right) \text{ (Catalan's constant),}$$

where

$$\left(\frac{-3}{\nu}\right) \equiv \nu \pmod{3} \quad \text{and} \quad \left(\frac{-4}{\nu}\right) \equiv \begin{cases} 0 & \text{for } \nu \text{ even,} \\ \nu \pmod{4} & \text{for } \nu \text{ odd} \end{cases}$$

are quadratic characters.

In spite of the complicated form of the recursions given in Sections 1–4 above, their solutions admit nice arithmetic properties if α is a rational number. For instance, the corresponding generating functions $U(z) = \sum_{n=0}^{\infty} u_n z^n$ satisfy the property $u(Az) \in \mathbb{Z}[[z]]$, where the integer A depends on $\alpha \in \mathbb{Q}$ (although a proof of the property in full generality for the recursions (4) and (10) is still beyond reach; see [10, 14, 20] for particular results). Beukers' computations [6] show that it is hard to expect modular parametrizations except in the above mentioned cases of the recursions for rational approximations to $\zeta(2)$ and $\zeta(3)$: the (Zariski closures of the) Galois groups associated to the linear differential operators annihilating functions $U(z)$ are richer than SL_2 . Beukers considers the recursion for rational approximations to $\zeta(4)$ (the case $\alpha = 0$ in (10)) and shows that the corresponding differential Galois group turns out to be O_5 , while the linear differential operator corresponding to the recursion for rational approximations to Catalan's constant (the case $\alpha = 1/2$ in (4)) is reducible. We expect that this differential reducibility holds for all $\alpha \notin \mathbb{Z}$ and that irreducible components of the corresponding differential operators are pullbacks of hypergeometric differential operators. The latter fact is closely related to the general conjecture (due to Dwork, Bombieri, ...) on the structure of differential G -operators. While chances to be able to attack this general conjecture seem to be small at the moment, to us it appears to be a nice and quite realistic program to give a direct proof in the case of the above recursions.

Acknowledgements. This work was done during a long-term visit at the Mathematical Institute of Cologne University. I thank the staff of the institute and personally P. Bundschuh for the hospitality and the warm working atmosphere. Special gratitude is due to the anonymous referee of the Journal of Computational and Applied Mathematics for the remarks and suggestions.

REFERENCES

1. R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$* , Astérisque **61** (1979), 11–13.
2. W.N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tracts, vol. 32, Cambridge Univ. Press, Cambridge, 1935; 2nd reprinted edition, Stechert-Hafner, New York–London, 1964.

3. F. Beukers, *A note on the irrationality of $\zeta(2)$ and $\zeta(3)$* , Bull. London Math. Soc. **11** (1979), no. 3, 268–272.
4. F. Beukers, *Irrationality proofs using modular forms*, Journées arithmétiques (Besançon, 1985), Astérisque **147–148** (1987), 271–283.
5. F. Beukers, *On Dwork’s accessory parameter problem*, Math. Z. **241** (2002), no. 2, 425–444.
6. F. Beukers, *Some Galois theory on Zudilin’s recursions*, The talk at the meeting on Elementary and Analytic Number Theory (Mathematisches Forschungsinstitut Oberwolfach, Germany, March 9–15, 2003).
7. F. Beukers and C. A. M. Peters, *A family of K3 surfaces and $\zeta(3)$* , J. Reine Angew. Math. **351** (1984), 42–54.
8. H. Cohen, *Accélération de la convergence de certaines récurrences linéaires*, Séminaire de Théorie des Nombres de Bordeaux (Année 1980–81), exposé 16, 2 pages.
9. L. A. Gutnik, *On the irrationality of certain quantities involving $\zeta(3)$* , Uspekhi Mat. Nauk [Russian Math. Surveys] **34** (1979), no. 3, 190; Acta Arith. **42** (1983), no. 3, 255–264.
10. C. Krattenthaler and T. Rivoal, *Hypergéométrie et fonction zêta de Riemann*, math.NT/0311114 (November 2003).
11. Yu. V. Nesterenko, *A few remarks on $\zeta(3)$* , Mat. Zametki [Math. Notes] **59** (1996), no. 6, 865–880.
12. M. Petkovšek, H. S. Wilf and D. Zeilberger, *A = B*, A. K. Peters, Ltd., Wellesley, M.A., 1996.
13. T. Rivoal, *La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 4, 267–270.
14. T. Rivoal, *Nombres d’Euler, approximants de Padé et constante de Catalan*, Ramanujan J. (to appear).
15. V. N. Sorokin, *One algorithm for fast calculation of π^4* , Preprint (April 2002), Russian Academy of Sciences, M. V. Keldysh Institute for Applied Mathematics, Moscow, 2002; available at <http://www.wis.kuleuven.ac.be/applied/intas/Art5.pdf>.
16. D. Zagier, *Integral solutions of Apéry-like recurrence equations*, Manuscript, 2003.
17. W. Zudilin, *Well-poised hypergeometric service for diophantine problems of zeta values*, Actes des 12èmes rencontres arithmétiques de Caen (June 29–30, 2001), J. Théorie Nombres Bordeaux **15** (2003), no. 2 (to appear).
18. W. Zudilin, *An elementary proof of Apéry’s theorem*, E-print math.NT/0202159 (February 2002).
19. W. Zudilin, *An Apéry-like difference equation for Catalan’s constant*, Electron. J. Combin. **10** (2003), no. 1, #R14.
20. W. Zudilin, *A few remarks on linear forms involving Catalan’s constant*, Chebyshevskii Sbornik (Tula State Pedagogical University) **3** (2002), no. 2 (4), 60–70; English transl., E-print math.NT/0210423 (October 2002).
21. W. Zudilin, *A third-order Apéry-like recursion for $\zeta(5)$* , Mat. Zametki [Math. Notes] **72** (2002), no. 5, 733–737.
22. W. Zudilin, *Well-poised hypergeometric transformations of Euler-type multiple integrals*, Preprint (April 2003), submitted for publication.

MOSCOW LOMONOSOV STATE UNIVERSITY
 DEPARTMENT OF MECHANICS AND MATHEMATICS
 VOROBIOVY GORY, GSP-2
 119992 MOSCOW, RUSSIA
 URL: <http://wain.mi.ras.ru/index.html>
 E-mail address: wadim@ips.ras.ru